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THE FREQUENCY OF X-RAY-INDUCED CHROMATID BREAKS IN TRADESCANTIA AS MODIFIED BY NEAR INFRARED RADIATION

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A modification of the frequency of x-ray-induced chromatid breaks can be achieved by varying the environmental conditions before, during or after x-radiation. All organisms, however, do not respond in like fashion to the same environmental changes. Sax and Enzmann[1] have shown that low temperatures (ca. 3°C.) effectively increase the frequency of x-ray-induced breaks in *Tradescantia;* high temperatures (ca. 38°C.) were shown to have the opposite effect. In *Drosophila,* temperature changes have been demonstrated to be ineffective in altering break frequencies,[2, 3] although earlier studies suggested that low temperatures can lead to an increase in x-ray-induced rearrangements.[4, 5]

The use of other types of radiation in combination with x-rays has opened a new avenue of attack on the problem of x-ray effects in that the results relative to chromosomal break frequencies do not appear to be additive. Ultra-violet radiation (2537 A), which can, itself, induce chromatid deletions,[6, 7] has been shown to produce a significant decrease in x-ray-induced rearrangements in both *Drosophila*[8] and *Tradescantia.*[9] Whether this can be attributed to an effect on the nucleic acid constituents of the nucleus remains to be determined. Near infrared radiation (<20,000 A), on the other hand, has quite the opposite effect. Ineffective in inducing breaks or rearrangements when used alone, and as detected by present cytological techniques, it considerably enhances the action of x-rays on the chromosomes of *Drosophila* if given before or during recombination of broken chromosome ends.[3, 10] Whether these data may be interpreted in terms of a local physical-chemical change resulting from a preferential absorption of energy by some chromosomal component, or in terms of a

more general cellular phenomenon, is as yet unknown. The present report deals with a preliminary study made of the modifying effect of near infrared radiation on x-ray-induced break and rearrangement frequencies in the microspore chromosomes of *Tradescantia*. The *Drosophila* results have been confirmed, and the investigation has been extended to other phases of the problem which are more readily studied in *Tradescantia*.

Material and Methods.—A clonal line of *Tradescantia paludosa*, Anders and Woodson, served as the experimental material for this study.[11]

The x-ray treatments were given by using a Coolidge-type tube with a tungsten target, operating at 170 kv. and 20 ma., and fitted with two filters, one of copper 0.25 mm. thick and the other of aluminum 1.06 mm. thick. At 40 cm., the distance from the target at which the inflorescences were treated, the dosage was 95 r units per minute, as measured by a Victoreen dosimeter. Only doses of one-half and one minute were used in the present study. All x-ray treatments were made by Mr. Henry Myers of the National Cancer Institute.

The method of applying the near infrared radiation was similar to that used by Kaufmann, Hollaender and Gay,[10] and described in greater detail by Hollaender, Kaufmann and Andrews.[12] The radiation from a 750-watt medium prefocus filament projection lamp was focused by means of a plano-convex glass lens through an iodine-saturated carbon tetrachloride filter $1^1/_2$ cm. deep and a Corning heat-resistant glass filter No. 242. The inflorescences were placed at the bottom of a large culture tube which rotated slowly in the focus of the infrared beam. The culture tube was cooled by running tap water during the exposure of seven hours in the infrared beam. (Preliminary experiments had shown that maintaining the inflorescences in running tap water (18–19°C.) either before or after x-radiation did not significantly alter the frequency of induced breaks and rearrangements.) An interval of 5–10 minutes usually elapsed between the end of one type of radiation and the beginning of the other.

The treated anthers were smeared, fixed in 3:1 alcohol-acetic and stained with aceto-carmine 22–23 hours after x-radiation. This time was selected because Sax[13] has shown that during the summer months chromosome, as opposed to chromatid, breaks appear at about 27 hours after x-raying; therefore, the appearance of chromosome breaks would have indicated that the rate of cell division had been speeded up by the application of infrared radiation. No chromosome breaks of any kind were observed, however, indicating that if infrared does speed up the rate of cell division it was counteracted by the cooling effect of the running tap water, or else it was not accelerated to any detectable degree.

The Effect of Pretreatment with Near Infrared Radiation.—Kaufmann, Hollaender and Gay[10] have shown that pretreatment with near infrared radiation significantly increases the frequency of detectable x-ray-induced

chromosomal rearrangements in *Drosophila* as compared to the frequency obtained from the x-ray controls. Dominant lethals, however, which supposedly result from inviable types of breaks, appeared not to be affected under the conditions of their experiments. In *Tradescantia*, where it is possible to detect most types of viable and inviable breaks and rearrangements, it was found that all detectable types of breaks were increased when infrared radiation was given prior to x-radiation (table 1). Single and double (isochromatid) deletions were increased in frequency approximately 100% over the x-ray controls when a dosage of 47.5 r units was used, while the exchange type of rearrangement, either within or between chromosomes, revealed a considerably greater increase in frequency. This was again found to be true when the x-ray dosage was doubled (95 r). It would appear, therefore, that rearrangements between chromosomes, or between arms of the same chromosome, were increased disproportionately more than single or double deletions when infrared pretreatment was given to x-rayed microspores, but when it is considered that the frequency of this type of rearrangement is a function of the square of the dosage while the frequency of single and double deletions is a linear function of the dosage,[13] the disproportionality of the exchange type of rearrangements in table 1 is readily explained. These data appear to be best interpreted in terms of a generalized "sensitization" of the chromosomes to x-rays rather than in terms of a differential effect on various types of breaks and rearrangements.[10]

TABLE 1

FREQUENCY OF CHROMATID ABERRATIONS INDUCED BY X-RAYS AND AS MODIFIED BY PRETREATMENT WITH NEAR INFRARED RADIATION; DATA IN %

TREATMENT	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL CHROM.	TOTAL BREAKAGE
47.5 r	2.86	1.72	0.29	2094	4.87 ± 0.46
Infrared + 47.5 r	5.63	2.68	1.72	1902	10.50 ± 0.70
Infrared + 47.5 r	5.02	3.40	2.72	2352	11.13 ± 0.65
95 r	6.27	5.89	2.15	1578	14.32 ± 0.88
Infrared + 95 r	11.48	9.84	11.88	1464	33.19 ± 1.23

The data presented in table 1 do not exclude from consideration the possibility that the increased break frequencies obtained when infrared pretreatment was combined with x-radiation might be related to a general temperature change induced within the cell. Measurements were therefore made in the test tube during the infrared treatment, and a fall of 2–3°C. below room temperature was recorded, the fall being due to the cooling effect of the running tap water. This was paralleled by a similar drop in temperature in the interior of the buds as determined by means of a thermocouple designed by Dr. J. Gordon Carlson. To be doubly certain, however, that the effects of infrared radiation were not related

to a general temperature effect (as opposed to a highly localized temperature change due to selective absorption), several groups of buds were given seven hours of pretreatment at various controlled temperatures before x-radiation, and then kept at room temperature for 22 hours, at which time they were smeared, fixed and stained.

TABLE 2

FREQUENCY OF X-RAY-INDUCED CHROMATID ABERRATIONS AS MODIFIED BY PRETREATMENT AT VARIOUS TEMPERATURES FOR SEVEN HOURS; 47.5 r; DATA IN %

TREATMENT	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL CHROM.	TOTAL BREAKAGE
6°C.	1.64	3.21	0.00	1464	4.85 ± 0.56
20	2.44	3.32	0.00	1680	5.77 ± 0.57
25 (room temp)	2.61	4.02	0.60	996	7.23 ± 0.82
34	1.34	3.03	0.99	2016	5.35 ± 0.51

The data are presented in table 2. The rate of breaks and rearrangements did not approach the level obtained when infrared radiation preceded x-raying, but the data indicate that a possible rise in exchange rearrangements may follow a rise in temperature, due possibly to a more actively dividing cell. That a generalized temperature change does not account for the modification of break frequencies as shown in table 1 is, however, evident.

TABLE 3

FREQUENCY OF CHROMATID ABERRATIONS INDUCED BY X-RAYS AND AS MODIFIED BY PRETREATMENT WITH INFRARED FILTERED THROUGH 11 CM. OF WATER; DATA IN %

TREATMENT	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL CHROM.	TOTAL BREAKAGE
47.5 r	2.47	2.17	0.59	2028	5.27 ± 0.50
Infrared + 47.5 r	5.36	3.32	2.78	1866	11.47 ± 0.74

Kaufmann, Hollaender and Gay[12] stated that the region of effectiveness in the infrared spectrum lies in the neighborhood of 10,000 A. This is the region most readily transmitted by the filter system which they used, and which is similar to that used in this study. However, some energy from longer wave-lengths up to 20,000 A was transmitted. In an effort to determine the effective region more exactly, the infrared radiation was filtered through a round flask 11 cm. in diameter filled with distilled water. No other filter was used. According to Brackett,[14] this filter system should absorb all wave-lengths longer than 11,500 A. The flask was water cooled to prevent it from becoming a secondary radiator. The inflorescences were irradiated as before for seven hours, and the data are contained in table 3. No significant deviations from the data in table 1 were obtained, indicating that wave-lengths longer than 11,500 A are relatively ineffective in modifying the frequency of breaks. The filter systems which have been used have thus restricted consideration to those wave-lengths which

lie between 6000 and 11,500 A. Further work is being carried out to determine the effectiveness of different wave-lengths within, as well as outside, this range.

The Effect of Post-treatment with Near Infrared Radiation.—Kaufmann, Hollaender and Gay[10] have shown that, in *Drosophila*, post-treatment of males with near infrared radiation led to a reduction in the frequency of detectable x-ray-induced chromosomal rearrangements, but this was subsequently interpreted as the result of a rapid maturation of sperm which were immature (and less sensitive to x-rays) when the flies were x-rayed, thus giving a heterogeneous sperm mixture at the time of copulation. Later Kaufmann[3] showed that the treatment of inseminated females with infrared radiation just prior to oviposition could modify the frequency of chromosomal changes in much the same manner as did pretreatment of males with infrared radiation.

TABLE 4

FREQUENCY OF CHROMATID ABERRATIONS INDUCED BY X-RAYS AND AS MODIFIED BY POST-TREATMENT WITH INFRARED RADIATION FOR SEVEN HOURS; DATA IN %

TREATMENT	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL CHROM.	TOTAL BREAKAGE
47.5 r	2.21	2.65	0.28	2898	5.15 ± 0.41
47.5 r + infrared	4.96	2.62	1.89	4656	9.47 ± 0.43

Since an interval of 22 hours existed between x-radiation and fixation of the chromosomes, there was an opportunity to alter the environmental conditions during the time when recombination is taking place. This was done with near infrared radiation, the length of treatment being seven hours. The data are given in table 4. The total frequency of chromatid changes was slightly lower than that obtained with pretreatments with infrared radiation, but an analysis of the individual types of changes indicated that the lowered frequency was due entirely to a lack of increase in the double deletions, the percentage remaining the same as that obtained from the x-ray controls. Single deletions and the exchange type of rearrangements were significantly increased, with the proportion of increase being comparable to that obtained with pretreatment of infrared radiation. From these data it may be inferred that double deletions are realized immediately on x-radiation, and hence are not subject to modification by post-treatment, whereas the type (or types) of break which gives rise to single deletions and exchange rearrangements is capable of remaining "open" for some time following x-radiation, and therefore its capacity for restitution or recombination can be readily altered by environmental conditions. These interpretations agree also with that obtained previously in *Tradescantia* when ultra-violet light, in combination with x-rays, served as the modifying influence.[9]

The Permanency of the Effect of Near Infrared Radiation.—The question

as to the permanency of the effect of infrared radiation on chromosome "sensitivity" can be readily answered merely by varying the time interval between pretreatments with infrared and x-radiation. This has been done in a very preliminary manner, but the data (table 5) are presented because they do yield some information relative to this aspect of the problem.

In this experiment a 250-watt, 120-volt reflector-drying lamp of the commercial variety was used as a source of near infrared, with the same filter system previously described being employed. Because of the greater output of transmitted energy, the time of exposure was reduced to approximately three hours. The infrared radiation was given prior to x-radiation, and the intervals of time between radiations were 16 and 21 hours, respectively. As with the other experiments, the treated anthers were smeared 22–23 hours after x-radiation.

TABLE 5

FREQUENCY OF X-RAY-INDUCED CHROMATID ABERRATIONS WHEN A DELAY EXISTS BETWEEN THE TIME OF NEAR INFRARED RADIATION AND TIME OF X-RADIATION; DATA IN %

TREATMENT	DELAY, HRS.	SINGLE DELETIONS	DOUBLE DELETIONS	EXCHANGES	TOTAL CHROM.	TOTAL BREAKAGE
47.5 r	..	1.69	1.36	0.136	1476	3.19 ± 0.46
Infrared + 47.5 r	0	4.42	3.47	1.77	1584	9.66 ± 0.74
Infrared + 47.5 r	16	4.89	3.83	1.89	1800	10.61 ± 0.73
Infrared + 47.5 r	21	4.39	3.73	0.88	1368	8.99 ± 0.77

The data obtained showed that the effect of near infrared radiation on the chromosomes was not dissipated within a short time, but rather that the "sensitization" of the chromosomes persisted for at least 21 hours, with all types of rearrangements showing a definite increase in frequency as compared with the x-ray controls. From these data it can be inferred that the effect of infrared radiation is not achieved by general temperature changes, or by a greater cell activity, but rather by a definite chemical or physical change within the cell which is reflected in the increased response of the chromosomes to x-radiation.

Discussion.—An intelligent discussion of the effects of near infrared radiation on the modification of the frequency of x-ray-induced chromosomal breaks and rearrangements is hindered and obscured by a lack of definite knowledge of the effects of this type of radiation on biological systems and processes. It is known that near infrared energies (1.2 electron volts for wave-lengths in the neighborhood of 10,000 A) are insufficient to produce a dissociation of molecular structures, differing in this respect from the ionizing action of the more penetrating forms of radiation, and the photochemical action of the shorter ultra-violet. That some change, however, is produced, and that this change, reflected in an increased breakability of the chromosomes when exposed to x-radiation, is not of momentary duration but one of relative permanency, and hence

probably of a physical-chemical nature, is evident from the data presented. The data, on the other hand, yield no evidence as to whether or not the effect on the chromosomes is direct or indirect, although it is logical to assume that it is the chromosome rather than some cytoplasmic component which is altered. Since absorption of the near infrared can lead to molecular vibration and rotation, the possibility remains that macromolecular systems such as are known to be present in chromosomes may undergo some structural change, resulting in a "sensitized" structure which is more readily disrupted by x-rays.[10] The concept of a sensitized structure, however, does not appear to explain satisfactorily the data obtained from post-treatment experiments since one must consider here the effect on breaks, potential or otherwise, which are already available for recombination. Post-treatment effects may be due to a mechanism which alters chromosome movement, as suggested by Kaufmann,[3] but most certainly the fact that a delay of 21 hours between pretreatment with infrared radiation and x-radiation does not materially alter the rate of increase in break frequency as compared to that when x-radiation immediately follows infrared treatment precludes any consideration of chromosome movement as an explanation for the data obtained from the pretreatment studies. The possibility of two distinct mechanisms therefore exists.

The following hypothesis suggests itself as an explanation by which both pre- and post-treatment data may be related to a single mechanism. If both infrared and x-rays are capable of weakening the chromosome structure (this in addition to the normally realized x-ray breaks) in a manner which would not be detectable as distinct breaks, and which would not be realized unless further disturbed by the addition of the other type of radiation, then an additional increase in break frequency would be expected with both pre- and post-treatments with infrared radiation. The failure to obtain an increase in the frequency of double deletions following post-treatment is readily explained by assuming that this type of breakage and recombination is realized immediately on x-radiation, and is thus not subject to modification by subsequent treatment.

Summary.—Experiments are reported in which it has been shown that near infrared radiation, when combined with x-rays, significantly increases the frequency of x-ray-induced breaks and rearrangements in the microspore chromosomes of *Tradescantia*. All types of detectable alterations are increased by pretreatment, while post-treatment increases the frequency of single deletions and exchanges but does not increase the frequency of double deletions. A delay of 21 hours between pretreatment with infrared and x-rays does not appreciably decrease the effectiveness of infrared, suggesting that the change induced by infrared is of a relatively permanent nature. The nature of the effect of infrared is poorly understood at the present time.

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REACTION AND SCATTERING CROSS-SECTIONS

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1. It was attempted, in two recent papers,[1] to give a formulation to the resonance theory of nuclear reactions[2] which is free from artificial assumptions. This theory deals with reactions in which two nuclei collide and either reseparate without any change (elastic scattering) or undergo a reaction so that the pair separating after the collision is different from the colliding pair (nuclear reaction or excitation). If the energy of the system does not make the second alternative possible, we shall speak of a one alternative reaction which is, in reality, elastic scattering. If the collision can yield, instead of the original pair of nuclei, another pair or two alternative pairs, we shall speak of a two or three alternative reaction. Thus, e.g., the collision of Be^9 and a neutron, which can yield either Be^9 + neutron, or $Li^8 + H^2$, or $He^6 + He^4$, or $Be^{10} + \gamma$-ray, will be called a four alternative reaction. The number of alternatives of the reaction which we investigate will be denoted by n. Contrary to what may seem from the above examples, this is usually not a very small number because most reactions can yield the end-products in numerous different states of excitation which are to be considered all as different alternatives. Reactions which yield more than two particles (e.g., $H^2 + H^1 = H^1 + H^2 +$ neutron) are excluded from the present treatment.

One arrives at a rather definite energy dependence of the cross-sections of reactions of the above nature if one assumes, as was done in the paper

of reference 1 (*a*), that the wave function of the compound nucleus (i.e., the wave function in that part of configuration space where all particles are close together) is, in first approximation and apart from an energy dependent coefficient, independent of energy. On the other hand, if one admits, as was done in 1 (*b*), that the wave function of the compound nucleus may be a linear combination of several, or even infinitely many, energy independent functions with energy dependent coefficients, the conclusions for cross-sections, etc., become much less definite. This is quite natural since this last assumption is in reality no assumption because every wave function, and hence also the wave function of the compound nucleus, can be written as a linear combination of a definite set of functions. Any specialization contained in the second of the above papers is contained in the discussions in which assumptions are made about the order of magnitude of the constants occurring in the expressions for cross-sections, etc. The most important assumption discussed is that the distance of the resonance levels from each other is greater than their width.

However, even if one does not make any such assumptions, the results of the second paper are not vacuous but contain at least three general statements. The first two of these are:

(i) That the elastic scattering cross-section does not vanish in general for any value of the energy, except in one alternative reactions.

(ii) The reaction cross-section vanishes for discrete values of the energy in two alternative reactions. In three and more alternative reactions no reaction cross-section (and because of (i) no scattering cross-section) vanishes in general for any value of the energy.

The third result is more difficult to formulate and will not be taken up at this occasion. However, a proof which is independent of the formalism employed in reference 1 will be given for the above assertions which will, at the same time, generalize them inasmuch as the assumption of zero angular momentum of the colliding particles will not be employed.

2. It is well known that the total effective cross-section of a reaction, such as the transformation of a pair j into a pair l, is a sum of several partial cross-sections even if all particles which participate in the reactions have zero spin. These partial cross-sections correspond to the different angular momenta $0, h, 2h, 3h, \ldots, Lh, \ldots$ of the colliding particles around their common center of mass:

$$\sigma_{jl} = \sum_L \sigma_{jl}^L. \tag{1}$$

The partial cross-sections can be expressed in terms of the matrix elements of the collision matrices U^L

$$\sigma_{jl}^L = \frac{\pi}{k_j^2}(2L+1)|U_{jl}^L - \delta_{jl}|^2. \tag{1a}$$

The collision matrices are n dimensional unitary symmetric matrices and they depend on the energy, k_j is the wave number corresponding to the relative motion of the colliding particles and δ_{jl} is zero except in the case of elastic scattering, i.e., if $j = l$. Equations (1) and (1a) are valid not only if the spins of all colliding particles and reaction products are zero but also if there is no interaction between orbital motion and spin. It follows from the conservation law for angular momentum in both cases that the relative angular momentum of the reaction products is equal to the relative angular momentum of the colliding particles.

The above two rules (i) and (ii) will be shown to hold for the partial cross-sections σ_{jl}^L rather than the total cross-section σ_{jl}. Since the σ_{jl}^L vanish, in general, only for isolated values of the energy, the σ_{jl} itself will not vanish, in general at all. However, as long as the total energy of the system is low, σ_{jl}^L for $L > 0$ will be very small and the vanishing of σ_{jl}^0 will entail the vanishing of σ_{jl} also. Even if several σ_{jl}^L contribute substantially to the total cross-section, the vanishing of one of them has some observable significance because the σ_{jl}^L can be separated to some extent due to the different angular distributions with which they are connected.

If some of the reaction products or colliding particles have spins and if these interact with the orbital motion, the situation is even more complicated and will not be described in detail. The total cross-sections will become sums of even more terms than (1) indicates. All the partial cross-sections will be connected with matrix elements of a collision matrix in a way similar to (1a) and the rules (i) and (ii) remain valid for the partial cross-sections. However, the decomposition of the total cross-section into partial cross-sections will be even more difficult experimentally than in the above-described simple case. It will probably be possible only for low energies when only particles colliding with zero angular momentum can react with each other.

3. The rules (i) and (ii) will be demonstrated now by the method of counting the number of free parameters (a) in a general symmetric unitary matrix, (b) in a symmetric unitary matrix the jj element of which is 1, and (c) in a symmetric unitary matrix of which an off diagonal element vanishes. First we prove that:

The characteristic vectors of a symmetric unitary matrix can be assumed to be real. From $U = U'$ follows $U^* = U^\dagger$ and from $U^\dagger U = 1$ then $U^* U = 1$. Hence from $Uv = \omega v$ follows $U^* U v = v = \omega U^* v$ or, taking the conjugate complex $\omega^* U v^* = v^*$. Since for the characteristic value ω of a unitary matrix $\omega^* = \omega^{-1}$, the last equation states that v^* is a characteristic vector of U and that it belongs to the same characteristic value to which v belongs. It then follows that the real vectors $v + v^*$ and $i(v^* - v)$, i.e., the real and imaginary parts of v, are also characteristic vectors of U so that the characteristic vectors can all be assumed to be real.

(As a matter of fact, the derivation of the symmetric nature of U shows that this nature is a consequence of the real character of its characteristic vectors.) As a result, U can be written in the form

$$U = R'\Omega R \tag{2}$$

where Ω is a diagonal matrix with diagonal elements of modulus 1 and R is a real orthogonal matrix. Conversely, $R'\Omega R$ is evidently both symmetric and unitary.

The number of real parameters in Ω is n, in R it is $^1/_2 n(n-1)$ so that U can be characterized by $^1/_2 n(n+1)$ real parameters.

Let us now consider a real symmetric matrix for which $U_{jj} = 1$. It then follows from the unitary condition that all U_{lj} and U_{jl} with $l \neq j$ vanish. As a result, the number of free parameters in an n dimensional matrix U with $U_{jj} = 1$ is the same as in a general $n-1$ dimensional U, i.e., $^1/_2(n-1)n$. One therefore has to fix $^1/_2(n+1)n - ^1/_2(n-1)n = n$ parameters to make an elastic scattering cross-section vanish. If one has only one parameter (the energy) at one's disposal, this will be possible only if $n = 1$, i.e., in one alternative reactions, that is, if, for reasons of energy, only elastic scattering is possible. This concludes the demonstration of (i).

When considering the case of a vanishing reaction cross-section, i.e., $U_{jl} = 0$ with $j \neq l$, we can assume, without loss of generality that $j = 1$, $l = 2$ so that $U_{12} = 0$.

If U is two-dimensional $U_{12} = 0$ means that U is a unitary diagonal matrix. Since the modulus of the diagonal elements must be 1, such a U will have two free parameters. The general two-dimensional symmetric unitary matrix has three free parameters so that it will be possible, in general, to make U_{12} vanish by a suitable choice of the energy. This concludes the discussion of the first part of (ii).

A three-dimensional symmetric unitary U with $U_{12} = 0$ has one of the two forms

$$\begin{Vmatrix} \omega\omega'^* \cos\varphi & 0 & \omega \sin\varphi \\ 0 & \omega'' & 0 \\ \omega \sin\varphi & 0 & -\omega\omega' \cos\varphi \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} \omega'' & 0 & 0 \\ 0 & \omega\omega'^* \cos\varphi & \omega \sin\varphi \\ 0 & \omega \sin\varphi & -\omega\omega' \cos\varphi \end{Vmatrix} \tag{3}$$

Since the modulus of all ω must be 1, both forms (3) contain four real parameters. The general three-dimensional symmetric unitary U has six parameters so that it will not be possible, in general, to make $U_{12} = 0$ by varying a single parameter. This demonstrates (ii) for three alternative reactions. (It also shows that if, for whatever reason, $U_{12} = 0$, either U_{23} or U_{13} must vanish also.)

If U has $n = 4$ or more dimensions, $U_{12} = 0$ can be expressed, by means of (2), as

$$U_{12} = \sum_j R_{j1}(\cos\varphi_j + i \sin\varphi_j)R_{j2} = 0, \tag{4}$$

where the diagonal elements of Ω were denoted by $\cos \varphi_j + i \sin \varphi_j$. As is well known, the number of real parameters in R, if (4) is not fulfilled, can be counted as follows: the first column R_{j1} is an n dimensional vector of length 1, giving $n - 1$ parameters; the second column R_{j2} is of length 1 and is orthogonal to the first column giving $n - 2$ parameters; the third column of R must be a unit vector orthogonal to the first and the second column, giving $n - 3$ parameters, etc. Altogether R has $n - 1 + (n - 2) + (n - 3) + \ldots + 1 = {}^1/_2(n - 1)n$ parameters. If (4) is fulfilled, the second column must be orthogonal to the first column and to the vectors with the components $R_{j1} \cos \varphi_j$ and $R_{j1} \sin \varphi_j$. It has, therefore, if these three vectors are linearly independent, only $n - 4$ parameters instead of the $n - 2$ as above. Hence, the condition $U_{12} = 0$ fixes two parameters and cannot be satisfied, in general, by varying the energy.

If only two of the vectors R_{j1}, $R_{j1} \cos \varphi_j$, $R_{j1} \sin \varphi_j$ are independent, R_{j2} still has to be orthogonal to one more vector than if the condition $U_{12} = 0$ is not imposed. In addition, the linear relation between the vectors R_{j1}, $R_{j1} \cos \varphi_j$, $R_{j1} \sin \varphi_j$ gives at least one additional condition so that the number of free parameters in U with $U_{12} = 0$ is, in this case, at least two smaller than it is in the general U. If both $R_{j1} \cos \varphi_j$ and $R_{j1} \sin \varphi_j$ are multiples of R_{j1}, all φ_j are equal which involves certainly more than two conditions. This shows that the conclusion of the preceding paragraph is valid in general and completes the demonstration of (ii).

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ALGEBRAIC MATRIC GROUPS

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Let $\mathfrak{G}$ be a multiplicative group of square matrices (a_{ij}) of degree $n \geqslant 1$ with elements a_{ij} contained in an algebraically closed field $\mathfrak{C}$. $\mathfrak{G}$ is called *algebraic* if there exists a set $\mathfrak{g}$ of polynomials in $\mathfrak{C}[\ldots, x_{ij}, \ldots]$ such that: (*a*) for each (a_{ij}) in $\mathfrak{G}$ and each $f(\ldots, x_{ij}, \ldots)$ in $\mathfrak{g}$ we have $f(\ldots, a_{ij}, \ldots) = 0$; (*b*) every non-singular matrix (a_{ij}), with elements in $\mathfrak{C}$ such that $f(\ldots, a_{ij}, \ldots) = 0$ for each $f(\ldots, x_{ij}, \ldots)$ in $\mathfrak{g}$, is in $\mathfrak{G}$. Such algebraic matric groups are encountered in the Picard-Vessiot theory of homogeneous linear ordinary differential equations, where they play a rôle similar to that played by finite permutation groups in the Galois theory of algebraic

equations. Any attempt to clarify and rigorize the Picard-Vessiot theory must include an adequate treatment of algebraic matric groups, and preferably (if the Picard-Vessiot theory is to be completely algebraic) an algebraic one, independent of the theory of Lie groups. The present communication describes some results along these lines. A description of an algebraic development of the Picard-Vessiot theory is contained in the note immediately following.

If we take $\mathfrak{g}$ as large as possible, so that it is unique, then $\mathfrak{G}$ consists of the algebraic manifold of $\mathfrak{g}$ (called the *underlying manifold* of $\mathfrak{G}$) with a lower-dimensional algebraic manifold deleted. It turns out that the irreducible components of the underlying manifold of $\mathfrak{G}$ are pairwise disjoint (except for singular matrices) and all have the same dimension, and the irreducible component containing the identity matrix ϵ is the underlying manifold of a normal algebraic subgroup of $\mathfrak{G}$ of finite index (= number of irreducible components). Definitions: this subgroup is the *component of the identity* of $\mathfrak{G}$ (notation: $\mathfrak{G}^\circ$); $\mathfrak{G}$ is *connected* if $\mathfrak{G} = \mathfrak{G}^\circ$; the *dimension* of $\mathfrak{G}$ is that of its underlying manifold.

We call $\mathfrak{G}$ *anticompact* if $\mathfrak{G}$ contains no matrix $\neq \epsilon$ of finite order not divisible by the field characteristic p, and call $\mathfrak{G}$ *quasicompact* if every algebraic subgroup of $\mathfrak{G}$ of order > 1 contains such a matrix. By making joint use of the algebraic manifold properties and group properties of $\mathfrak{G}$ we can prove: (1) *$\mathfrak{G}$ is anticompact if and only if each matrix of $\mathfrak{G}$ is reducible to special triangular form* (0's below the main diagonal, 1's on it), i.e., if and only if each matrix in $\mathfrak{G}$ has all its characteristic roots equal to 1. (2) *$\mathfrak{G}$ is quasicompact if and only if each matrix in $\mathfrak{G}$ is reducible to diagonal form.*

$\mathfrak{G}$ is called *solvable* if $\mathfrak{G}$ has a normal chain in which all the factor groups are abelian (here "normal chain" signifies a normal chain in the usual sense plus the restriction that all members of the chain be algebraic). $\mathfrak{G}$ is solvable if and only if its sequence of commutator subgroups terminates with the identity group (the commutator subgroup $\mathfrak{G}'$ of $\mathfrak{G}$ is the smallest algebraic subgroup of $\mathfrak{G}$ containing $\sigma\tau\sigma^{-1}\tau^{-1}$ for all σ and τ in $\mathfrak{G}$). Using several lemmas it is possible to prove that *if $\mathfrak{G}$ is connected and solvable then $\mathfrak{G}$ is reducible to triangular form* (0's below the main diagonal). The proof employs a double induction on the matrix degree n and the normal chain length r. For $n = 1$ there is nothing to prove. Letting $n > 1$ and supposing the result verified for matrices of lower degree, we can assume that $\mathfrak{G}$ is irreducible (this requires a lemma asserting that the blocks of a reduced algebraic matric group are themselves algebraic matric groups, connected when the given reduced group is). For $r = 1$ (i.e., for abelian $\mathfrak{G}$) the result is an easy consequence of Schur's lemma. Letting $r > 1$ and supposing the result verified for matrices of degree n with lower values of r, we see that the second member $\mathfrak{G}_1$ of the normal chain for $\mathfrak{G}$ of length

r is reducible to triangular form (this requires a lemma asserting that $\mathfrak{G}'$ is connected whenever $\mathfrak{G}$ is, permitting the assumption that $\mathfrak{G}_1$ is connected). Then an argument making use of the reducibility of $\mathfrak{G}_1$ to triangular form and the abelian nature of $\mathfrak{G}/\mathfrak{G}_1$ leads to a contradiction.

Using this theorem and the results 1 and 2 above, it can be shown that *$\mathfrak{G}$ is solvable and anticompact if and only if $\mathfrak{G}$ is reducible to special triangular form*, and *if $\mathfrak{G}$ is reducible to triangular form and is quasicompact then $\mathfrak{G}$ is reducible to diagonal form.*

Finally, we investigate the extent to which the concepts "solvable," "anticompact," and "quasicompact" are broadened by the introduction of apparently more inclusive definitions by means of conditions on a normal chain

$$\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \ldots \supseteq \mathfrak{G}_{r-1} \supseteq \mathfrak{G}_r = \mathfrak{E}.$$

Using above-mentioned results concerning $\mathfrak{G}^\circ$ and $\mathfrak{G}'$ it is not hard to show that *if every factor group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ is abelian or finite then $\mathfrak{G}^\circ$ is solvable.*

If the definitions of anticompact and quasicompact are extended to factor groups in the obvious way, it can be proved (using a slight generalization of result 1 above) that: *if every factor group $\mathfrak{G}_{i-1}/\mathfrak{G}_1$ is anticompact or finite then $\mathfrak{G}^\circ$ is anticompact; if every $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ is anticompact then $\mathfrak{G}$ is, too.* Analogous to the second part of this theorem is the result that *if every $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ is quasicompact; then $\mathfrak{G}$ is, too.* I do not know whether the analog to the first part is true or false (for $p = 0$ it is obviously true, as a finite group is then quasicompact; for $p > 0$ a finite group may not be quasicompact, e.g., a group of order divisible by p).

THE PICARD-VESSIOT THEORY OF HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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The Galois theory of homogeneous linear ordinary differential equations as developed by Picard and Vessiot is founded on the theory of Lie groups and on the general theory of differential equations.[1] Because of the loose state of the Lie theory at the time, the weakness then of the theory of differential equations with respect to its algebraic aspects, and the overintimate connection with the analytic point of view, the Picard-Vessiot theory suffers from a certain lack of rigor, completeness and simplicity. The present communication describes an attempt to algebraize, rigorize, round out and extend the Picard-Vessiot theory. Use is made of the Ritt

theory of algebraic differential equations[2] (unavailable, of course, to Picard and Vessiot), and some results concerning algebraic groups of matrices developed for the purpose.[3]

Let $\mathfrak{F}$ be a differential field (ordinary or partial) of characteristic 0, $\mathfrak{G}$ a differential extension field thereof. A set of isomorphisms of $\mathfrak{G}$ over $\mathfrak{F}$ (i.e., isomorphisms of $\mathfrak{G}$ under which each element of $\mathfrak{F}$ is invariant) will be called *abundant* if for each differential field $\mathfrak{F}_1$ between $\mathfrak{F}$ and $\mathfrak{G}$ and each element α in $\mathfrak{G} - \mathfrak{F}_1$ there is an isomorphism in the set under which α is not invariant but every elment of $\mathfrak{F}_1$ is. By a previous paper[4] such sets of isomorphisms always exist. $\mathfrak{G}$ will be called a *normal* extension of $\mathfrak{F}$ if the group of all automorphisms of $\mathfrak{G}$ over $\mathfrak{F}$ is abundant.

Let $\mathfrak{G}$ be a normal extension of $\mathfrak{F}$, and let $\mathfrak{H}$ be an abundant group of automorphisms of $\mathfrak{G}$ over $\mathfrak{F}$ (not necessarily the group of all such automorphisms). For any differential field $\mathfrak{F}_1$ between $\mathfrak{F}$ and $\mathfrak{G}$ let $\mathfrak{H}(\mathfrak{F}_1)$ denote the group of all automorphisms in $\mathfrak{H}$ which leave invariant each element of $\mathfrak{F}_1$ (thus $\mathfrak{H}(\mathfrak{F}) = \mathfrak{H}$ and $\mathfrak{H}(\mathfrak{G}) = \mathfrak{E}$, the identity group). Then it is easy to show that *the mapping* $\mathfrak{F}_1 \to \mathfrak{H}(\mathfrak{F}_1)$ *is a one-to-one correspondence between the set of all differential fields between* $\mathfrak{F}$ *and* $\mathfrak{G}$ *and a certain set of subgroups of* $\mathfrak{H}$. Furthermore: $\mathfrak{H}(\mathfrak{F}_1)$ *is a normal subgroup of* $\mathfrak{H}$ *if and only if* $\sigma\mathfrak{F}_1 = \mathfrak{F}_1$ *for every* σ *in* $\mathfrak{H}$; *when this condition is satisfied then* $\mathfrak{F}_1$ *is a normal extension of* $\mathfrak{F}$ *and* $\mathfrak{H}/\mathfrak{H}(\mathfrak{F}_1)$ *is isomorphic with an abundant group of automorphisms of* $\mathfrak{F}_1$ *over* $\mathfrak{F}$. I do not know, in the case in which $\mathfrak{H}$ is the group of all automorphisms of $\mathfrak{G}$ over $\mathfrak{F}$, whether $\mathfrak{H}/\mathfrak{H}(\mathfrak{F}_1)$ is isomorphic with the group of all automorphisms of $\mathfrak{F}_1$ over $\mathfrak{F}$.

Henceforth, suppose that $\mathfrak{F}$ is an ordinary differential field of characteristic 0 with an algebraically closed field of constants $\mathfrak{C}$, and consider a homogeneous linear differential polynomial $L(y) = y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y$ (each p_i in $\mathfrak{F}$). If $\eta_1, \ldots, \eta_n$ are n solutions of $L(y) = 0$, linearly independent over $\mathfrak{C}$, which are contained in some extension of $\mathfrak{F}$, and if the differential field $\mathfrak{G}$ obtained by adjoining $\eta_1, \ldots, \eta_n$ to $\mathfrak{F}$ contains no constants not in $\mathfrak{C}$, then $\mathfrak{G}$ will be called a *Picard-Vessiot* extension of $\mathfrak{F}$. It turns out that *every Picard-Vessiot extension* $\mathfrak{G}$ *of* $\mathfrak{F}$ *is normal, and the group* $\mathfrak{H}$ *of all automorphisms* σ *of* $\mathfrak{G}$ *over* $\mathfrak{F}$ *is (isomorphic with) an algebraic group (also denoted by* $\mathfrak{H}$*) of matrices* (k_{ij}), *with each* k_{ij} *in* $\mathfrak{C}$, *such that* $\sigma\eta_j = \sum_{i=1}^{n} k_{ij}\eta_i$ $(j = 1, \ldots, n)$. For the special case of Picard-Vessiot extensions it can be proved that the set of all groups $\mathfrak{H}(\mathfrak{F}_1)$ with $\mathfrak{F}_1$ between $\mathfrak{F}$ and $\mathfrak{G}$ is identical with the set of *all* algebraic subgroups of $\mathfrak{H}$; that when $\mathfrak{H}(\mathfrak{F}_1)$ is a normal subgroup of $\mathfrak{H}$ then $\mathfrak{H}/\mathfrak{H}(\mathfrak{F}_1)$ is isomorphic with the group of *all* automorphisms of $\mathfrak{F}_1$ over $\mathfrak{F}$, and that the *dimension* of the algebraic matric group $\mathfrak{H}$ equals the *degree of transcendency* of $\mathfrak{G}$ over $\mathfrak{F}$.

An element α of an extension of $\mathfrak{F}$ is called an *integral* of an element a of that extension if $\alpha' = a$; α is called an *exponential of an integral* of a if $\alpha' = a\alpha$. A differential extension field $\mathfrak{K}$ of $\mathfrak{F}$ will be called *liouvillian*

if: (*a*) every constant in $\mathfrak{K}$ is in $\mathfrak{C}$; (*b*) $\mathfrak{C}$ is an extension of $\mathfrak{F}$ by means of integrals, exponentials of integrals, and algebraic functions, i.e., there is a monotonic sequence of differential fields $\mathfrak{F} = \mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \ldots \subseteq \mathfrak{F}_r = \mathfrak{K}$ such that for each $i > 0$ $\mathfrak{F}_i$ is obtained from $\mathfrak{F}_{i-1}$ by the differential field adjunction of a single element which is either an integral of an element of $\mathfrak{F}_{i-1}$, an exponential of an integral of an element of $\mathfrak{F}_{i-1}$, or algebraic over $\mathfrak{F}_{i-1}$.

We shall have occasion to distinguish ten types of liouvillian extension, namely, extension by:

(1) integral, exponentials of integrals, and algebraic functions (i.e., any liouvillian extension),
(2) integrals and exponentials of integrals,
(3) exponentials of integrals, and algebraic functions,
(4) integrals and algebraic functions,
(5) integrals and radicals,
(6) exponentials of integrals,
(7) integrals,
(8) algebraic functions,
(9) radicals,
(10) rational functions (i.e., not a proper extension at all).

Corresponding to these ten types of liouvillian extension we consider ten types of algebraic matric groups defined by properties of the component of the identity $\mathfrak{G}^\circ$ and of $\mathfrak{G}$ itself:

(1) $\mathfrak{G}^\circ$ is solvable,
(2) $\mathfrak{G}$ is solvable,
(3) $\mathfrak{G}^\circ$ is solvable and quasicompact,
(4) $\mathfrak{G}^\circ$ is solvable and anticompact,
(5) $\mathfrak{G}$ is solvable and $\mathfrak{G}^\circ$ is anticompact,
(6) $\mathfrak{G}$ is solvable and quasicompact,
(7) $\mathfrak{G}$ is solvable and anticompact,
(8) $\mathfrak{G}$ is finite,
(9) $\mathfrak{G}$ is finite and solvable,
(10) $\mathfrak{G} = \mathfrak{E}$.

It is now possible to state the following extension of Vessiot's big theorem on "solvability by quadratures." *If the Picard-Vessiot extension $\mathfrak{G}$ is contained in a liouvillian extension of $\mathfrak{F}$ then $\mathfrak{G}^\circ$ is solvable. Conversely, if $\mathfrak{G}^\circ$ is solvable then $\mathfrak{G}$ is a liouvillian extension of $\mathfrak{F}$. In either case the liouvillian extension is of one of the types (1)–(10) if and only if the algebraic matric group $\mathfrak{G}$ is of the corresponding type (1)–(10).*

[1] For the literature of the Picard-Vessiot theory see: Vessiot, E., *Encyclopédie des sciences mathématiques pures et appliquées*, tome II, vol. 3, fascicule 1, 58–170, esp. pp. 152–165 (1910).

[2] A general account of this theory as of 1938 is contained in: Ritt, J. F., *American Mathematical Society Semicentennial Publications*, II, 35–55 (1938).

[3] See the immediately preceding note. Familiarity with the results of that note will be assumed in the present communication.

[4] Kolchin, E. R., *Annals of Mathematics*, **43**, 724–729 (1942).

A CHARACTERIZATION OF A SIMPLE PLANE WEB

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The compact continuum M is said to be a simple web if there exist an upper semicontinuous collection G_1 of mutually exclusive continua, and another such collection G_2, such that (1) each of these collections fills up M, (2) each of them is a dendron with respect to its elements and (3) if g_1 is a continuum of G_1 and g_2 is a continuum of G_2, the common part of g_1 and g_2 exists and is totally disconnected.

In his thesis,[1] R. H. Bing has shown that in order that the compact plane continuum M should be a simple web it is necessary and sufficient that it should be a continuous curve which remains connected and locally connected on the removal of any countable point set. In the present paper[2] another characterization will be given.

THEOREM 1. *Every two points of a simple plane web are separated from each other in it either by each of uncountably many mutually exclusive arcs or by each of uncountably many mutually exclusive simple closed curves.*

Proof. Suppose M is a simple web in the plane E. The point set M is[3] a continuous curve and[4] the boundaries of its complementary domains are simple closed curves no two of which have more than one point in common. Bing introduces and makes extensive use of the notion of a subcontinuum of M which is maximal with respect to being the closure of a connected point set which is the sum of a countable number of points and boundaries of complementary domains of M. Let H denote the set of all such continua. Every element of H is a continuous curve every non-degenerate cyclic element of which is a simple closed curve bounding some complementary domain of M and the collection H is contracting. Let G denote the collection whose elements are the continua of H and the points of M that belong to no continuum of H. Suppose A and B are points of M.

Suppose first that A and B belong to the same continuum h of the collection H. There exists a non-degenerate cyclic element J of h such that if h-J exists no component of its closure contains both A and B. Let W denote the collection of all continua w such that w is either (1) a con-

tinuum of the collection G distinct from h, (2) a component of the closure of h-J, (3) a point of J which is not a limit point of any component of h-J or (4) a point of D, the complementary domain of M of which J is the boundary. The collection W is upper semicontinuous and there exists a reversible transformation T throwing the continua of W into the points of a spherical surface S, or into those of a plane S, and such that (1) if J' denotes the set of all continua of the collection W that intersect J, $T(J')$ is a circle and (2) T is reversibly continuous in the sense that the sequence $w_1, w_2, w_3, \ldots$ of continua of the collection W converges to a subcontinuum of the continuum w of W if and only if the sequence $T(w_1), T(w_2), T(w_3), \ldots$ of points of S converges to the point $T(w)$ of S. Let A' and B' denote the continua of the collection W that contain A and B, respectively. In S the point $T(A')$ belongs to the circle $T(J')$. Let Z denote the set of all circles lying on S, with center at $T(A')$ and with radii less than the distance from $T(A')$ to $T(B')$. For each circle z of the collection Z, let α_z denote the arc of z which contains no point of $T(D)$ but has, as its end-points, two points of $T(J')$. Let Q denote the collection of all such arcs α_z for all circles z of the collection Z. Each arc of the collection Q separates $T(A')$ from $T(B')$ in $S - T(D)$. But the collection Q is uncountable and the set of all non-degenerate continua whose images under T are points of S is countable and thus all but a countable number of the arcs of Q are images under T of arcs in the plane E. But since every arc of Q separates $T(A')$ from $T(B')$ in $S - T(D)$ therefore A is separated from B in M by every arc of which an arc of Q is the image under T. It follows that there are uncountably many mutually exclusive arcs each separating A from B in M.

Suppose now that no continuum of the collection H contains both A and B. In this case there exists a transformation T throwing the continua of G into the points of a sphere S and such that T is reversibly continuous in the sense described above. Let A' and B' denote the continua of G that contain A and B, respectively. On the spherical surface S, $T(A')$ is separated from $T(B')$ by every circle on S with center at $T(A)$ and radius less than the distance from $T(A')$ to $T(B')$. All but a countable number of these circles are images under T of simple closed curves and every one of these simple closed curves separates A from B in M.

THEOREM 2. *If every two points of the compact continuum M are separated from each other in M by each of uncountably many mutually exclusive subcontinua of M but M has no cut point then M is not separated by any countable point set.*

Proof. Since every two points of M are separated from each other in M by a continuum, M is[5] a continuous curve. Suppose the points A and B are separated from each other in M by a countable point set K. Then there exists a closed non-degenerate subset H of K such that H separates

A from B in M but no proper subset of H does so. Every point of H is a limit point of both M_A and M_B, the components of $M - H$ that contain A and B, respectively. By hypothesis, H is non-degenerate. Let E and F denote two of its points. There exists a subcontinuum L of M separating E from F in M and containing no point of K. Since $E + F + M_A$ and $E + F + M_B$ are connected subsets of M, L intersects both M_A and M_B. Hence $L + M_A + M_B$ is connected. But it contains A and B and no point of H. This involves a contradiction.

THEOREM 3. *If every two points of the compact continuum M are separated from each other in M by each of uncountably many mutually exclusive subcontinua of M and M has no cut point then if e is a positive number there do not exist in M infinitely many mutually exclusive arcs of diameter more than e such that each of them is the sum of a countable number of continua t such that t is either a point or a subset of the boundary*[6] *of some complementary domain of M.*

Proof. Suppose that, for some e, M contains infinitely many mutually exclusive arcs satisfying these conditions. Then there exists a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ of them converging to some non-degenerate continuum N. Let E and F denote two points of N. There exists a subcontinuum T of M separating E from F in M and such that, if T intersects α_n, every component of $T.\alpha_n$ is a subset of some segment of α_n lying on the boundary of some complementary domain of M. There exist circles J_E and J_F with centers at E and F, respectively, and a sequence of mutually exclusive arcs $E_1F_1, E_2F_2, \ldots$ such that, for each n, E_nF_n is, for some m, an interval of α_m lying wholly without each of the circles J_E and J_F, except that E_n and F_n belong to J_E and J_F, respectively, and furthermore T intersects every arc of the sequence $E_1F_1, E_2F_2, \ldots$ and lies wholly without both J_E and J_F. There exist two integers i and j, less than or equal to 3, and arcs E_iE_j and F_iF_j lying on J_E and J_F, respectively, such that T intersects both I, the interior and E, the exterior, of J, the simple closed curve $E_iF_i + F_iF_j + E_jF_j + E_iE_j$. It follows that some component Q of $T.J$, and therefore either of $T.(E_iF_i)$ or of $T.(E_jF_j)$, contains both a limit point of of $T.I$ and a limit point of $T.E$. But Q is a subset of a segment of E_iF_i or of E_jF_j lying on the boundary of some complementary domain D of M. The domain D contains a point of M. This involves a contradiction.

THEOREM 4. *If every two points of the compact continuum M are separated from each other in M by each of uncountably many mutually exclusive subcontinua of M but M has no cut point then if K is a countable subset of M, $M - K$ is locally connected.*

Proof. A point set will be said to be of Class 1 if it is the sum of a countable number of continua such that each of them is either a point or a subset of the boundary of some complementary domain of M.

For each point P of M let M_P denote the point set consisting of P and

all points X of M, if there are any, such that X and P are the extremities of an arc of Class 1.

Suppose P is a point of M. Suppose A is a limit point of M_P not belonging to it. There exists a sequence $P_1, P_2, P_3, \ldots$ of distinct points of $M_P - P$ converging to A. For each n there exists an arc PP_n of Class 1. Let n_1 denote 1, let B_{n_1} denote P and let n_2 denote the smallest positive n such that P_m does not belong to PP_{n_1} for any m greater than or equal to n. Let B_{n_2} denote the first point in the order from P_{n_2} to P that PP_{n_2} has in common with PP_{n_1} and let $B_{n_2}P_{n_2}$ denote the interval of PP_{n_2} whose end-points are B_{n_2} and P_{n_2}. There exist an infinite sequence of positive integers $n_1, n_2, n_3, \ldots$, an infinite sequence of points $B_{n_1}, B_{n_2}, B_{n_3}, \ldots$ and an infinite sequence of arcs $B_{n_1}P_{n_1}, B_{n_2}P_{n_2}, \ldots$ such that (1) B_{n_1}, B_{n_2} are as described, (2) for each j, n_{j+1} is the smallest positive integer n such that P_m does not belong to PP_{n_i} for any m greater than or equal to n, and i less than or equal to j, (3) $B_{n_{j+1}}$ is the first point, in the order from $P_{n_{j+1}}$ to P on $PP_{n_{j+1}}$ that $PP_{n_{j+1}}$ has in common with the point set $B_{n_1}P_{n_1} + B_{n_2}P_{n_2} + \ldots + B_{n_j}P_{n_j}$ and (4) $B_{n_{j+1}}P_{n_{j+1}}$ is the interval of $PP_{n_{j+1}}$ whose end-points are $B_{n_{j+1}}$ and $P_{n_{j+1}}$. No two of the arcs $B_{n_1}P_{n_1}, B_{n_2}P_{n_2}, B_{n_3}P_{n_3}, \ldots$ have more than one point in common. Suppose there exists a positive number e such that, for infinitely many integers i, the diameter of $B_{n_i}P_{n_i}$ is greater than e. Then for each i there exists an interval of $B_{n_i}P_{n_i}$ of diameter more than e and having neither B_{n_i} nor P_{n_i} as an end-point. This is contrary to Theorem 3. It follows that no arc of the sequence $B_{n_1}P_{n_1}, B_{n_2}P_{n_2}, \ldots$ intersects more than a finite number of the others. Hence there exists an infinite sequence $Q_1, Q_2, Q_3, \ldots$ such that (1) for each n, Q_n is a finite collection of one or more arcs of the sequence $B_{n_1}P_{n_1}, B_{n_2}P_{n_2}, \ldots$, (2) $B_{n_1}P_{n_1}$ is the only element of Q_1, (3) each arc of Q_{n+1} intersects some arc of Q_n, but intersects no arc of Q_{n-1} if $n > 1$, (4) each arc of the sequence $B_{n_1}P_{n_1}, B_{n_2}P_{n_2}, \ldots$ belongs to some collection of the sequence $Q_1, Q_2, Q_3, \ldots$. There exists[7] an infinite sequence $m_1, m_2, m_3, \ldots$ of distinct integers of the sequence $n_1, n_2, n_3, \ldots$ such that, for each j, $B_{m_j}P_{m_j}$ belongs to Q_j and intersects $B_{m_{j+1}}P_{m_{j+1}}$. If, for each j, α_j denotes the point B_{m_j} or the interval $B_{m_j}B_{m_{j+1}}$ of $B_{m_j}P_{m_j}$ according as B_{m_j} is or is not identical with $B_{m_{j+1}}$, then $A + \alpha_1 + \alpha_2 + \ldots$ is an arc from P to A. But it is of Class 1. Hence A belongs to M_P and M_P is closed.

With the aid of Theorem 3 and the fact that every two points of M_P are the extremities of an arc of Class 1 lying in M_P, it may be shown[8] that if A is a point of M_P and ϵ is a positive number there exists a positive number δ_ϵ such that every point of M_P at a distance less than δ_ϵ from A lies together with A in an arc of Class 1 of diameter less than ϵ. It follows that not only is M_P a continuous curve but every two points of a connected open subset of M_P are the extremities of an arc of Class 1 lying in that subset.

Suppose AXB is an arc of Class 1 lying in M_P and suppose no boundary

of a complementary domain of M contains a segment of AXB containing X. Suppose X does not separate A from B in M_P. Then, since M_P is a continuous curve, if D is the component of $M_P - X$ that contains A, D is an open subset of $M_P - X$ containing B and therefore it contains an arc AYB of Class 1. The point set $AXB + AYB$ contains a simple closed curve J of Class 1 containing a segment of AXB containing X. The curve J is not the boundary of any complementary domain of M. But it is the sum of two mutually exclusive point sets N_1 and N_2 such that N_1 is countable and N_2 is the sum of two or more segments each of which is a subset of some complementary domain of M. If I and E denote the interior and exterior of J the point sets $M.I$ and $M.E$ exist and N_1 separates them from each other in M. But this is contrary to Theorem 2.

If A and B are points of M_P and the boundary of no complementary domain of M contains both of them there exists an arc ACB of Class 1 lying in M_P and there exists a cut point X of ACB belonging to no segment of ACB which is a subset of the boundary of a complementary domain of M. Furthermore if a point X of M_P is common to J_1 and J_2, the boundaries of two different complementary domains of M, and A and B are points of J_1 and J_2, respectively, distinct from X, and AX and BX are arcs lying on J_1 and J_2, respectively, then $AX + BX$ is an arc of Class 1 and the point X belongs to no segment of AXB which is a subset of any complementary domain of M. It follows that (1) if two points of M_P do not belong to the boundary of the same complementary domain of M they are separated from each other in M_P by some point and (2) if the boundaries of two complementary domains of M have a point of M_P in common that point separates M_P. Furthermore if the boundary of a complementary domain of M intersects M_P it is a subset of M_P. Hence every cyclic element of M_P is a simple closed curve which is the boundary of some complementary domain of M.

Let G denote the collection of all continua g such that, for some point P of M, g is M_P. The continua of the collection G are mutually exclusive and if ϵ is a positive number there are not more than a finite number of them of diameter more than ϵ, and no continuum of G separates any two continua of G from each other in the plane. Furthermore G fills up M and the boundary of each complementary domain of M is a subset of some continuum of G. Hence G is upper semicontinuous and with respect to its elements regarded as points, G is topologically equivalent to the surface of a sphere. With the help of this fact and the fact that the surface of a sphere remains connected and locally connected on the removal of a countable point set, it may be shown that if K is a countable subset of M and H is the set of all elements of G that intersect K then (1) $M-H^{*9}$ is connected and (2) it is locally connected at every one of its points not belonging to a non-degenerate continuum of G. But every point of a

continuum of H is a limit point of $M - H^*$. Hence, $M - K$ is locally connected at every point of $M - K$ belonging to no non-degenerate continuum of G. Suppose now that P is a point of $M - K$ belonging to the non-degenerate continuum g of the collection G. If L denotes the point set $(M - g) + P$ and N denotes $L - K \cdot (M - g)$ then both L and N are locally connected at P and every point of g that does not belong to K is a limit point of N. Hence $M - K$ is locally connected at P. Thus the omission of any countable point set leaves M locally connected.

The following theorem has now been established.

THEOREM 5. *In order that the compact non-degenerate plane continuum M with no cut point should be a continuous curve which remains both connected and locally connected on the removal of any countable point set, it is necessary and sufficient that for every two points X and Y of M there should exist uncountably many mutually exclusive continua each separating X from Y in M.*

[1] Bing, R. H., "Concerning Simple Plane Webs," *Trans. Am. Math. Soc.*, **60**, 133–148 (1946).

[2] In this paper, except where there is some indication to the contrary, it will be understood that space is a Euclidean plane E.

[3] Moore, R. L., "Concerning Webs in the Plane," these PROCEEDINGS, **29**, 389–393 (1943).

[4] Bing, R. H., *loc. cit.*

[5] Moore, R. L., "A Characterization of a Continuous Curve," *Fundamenta Mathematicae*, **VII**, 302–307 (1925).

[6] Since the continuous curve M has no cut point, therefore, by Theorem 9 of my paper "Concerning the Common Boundary of two Domains," *Ibid.*, **VI**, 203–213 (1924), the boundary of every complementary domain of M is a simple closed curve. Since M is not separated by any pair of its points, no two of these boundaries have more than one point in common. Cf. R. H. Bing, *loc. cit.*

[7] This may be established by an argument strictly analogous to that given to prove Theorem 78 in Chapter I, of my book "Foundations of Point Set Theory," *Am. Math. Soc. Colloquium Pub.*, Vol. XIII, New York (1932).

[8] This may be done with the aid of an argument having much in common with the argument given to prove Theorem 8 in Chapter II of the book cited in footnote 7.

[9] The notation H^* is used to denote the sum of all the point sets of the collection H.

CYCLOTOMY AND TRINOMIAL EQUATIONS IN A FINITE FIELD

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H. H. Mitchell[1] considered a finite field of order q^t where q is any prime and $q^t - 1 = e\nu$. Since the group formed under multiplication by the non-zero elements of the field is cyclic we may represent all such elements as powers of one element, say g. Denote the number of solutions of the congruence

$$1 + \sigma_i \equiv \sigma_j \tag{1}$$

by $[i, j]$ where σ_i denotes any element whose index is congruent to i modulo e.

Represent also the residue of index (-1), modulo e, by ϵ, so that if q is odd $\epsilon \equiv 0, e/2$ according as ν is even or odd, whereas if $q = 2$, $\epsilon \equiv 0$.

He then found

$$\sum_i [i, 0] = \nu - 1; \quad \sum_j [i, j] = \nu \tag{2}$$

and also the following quadratic relations

$$\sum_i [i, j][k - i, m - i] = \sum_i [i, k][j - i, m - i], \tag{3}$$

where $i \equiv 0, 1, 2, \ldots, e - 1$; $j \equiv 1, 2, \ldots, e - 1 \pmod e$, whereas if $j \equiv 0$, $m \equiv k \pmod e$, we must add ν to the left-hand side of the equation, and if $k \equiv 0$, $m \equiv j \pmod e$, we must add ν to the right-hand side.

The equation (1) may also be written in the form

$$1 + g^{i+re} = g^{j+se} \tag{4}$$

and we may note that $[i, j]$ is the number of solutions g^r, g^s, of (4), if r, $s = 0, 1, 2, \ldots, \dfrac{q^t - 1}{e} - 1$. In other papers[2] the writer considered the number of solutions g^w, g^w, of

$$1 + g^{f+we_1} = g^{h+we_2}. \tag{5}$$

If we denote the number of solutions of (5) by (f, h), then we shall show in another paper that quadratic relations, of a bit different type from (3), exist involving such symbols. In the present paper we develop some formulae which we shall need in the later paper. For this purpose we shall employ some of the principles of the theory of cyclotomic fields.

Let p be an odd prime belonging to the exponent t modulo l and such

that $p^t = 1 + cl$ with l an odd prime and $(c, l) = 1$. Write $\theta = \zeta\beta$, where ζ is a primitive lth root of unity and β is a primitive cth root of unity. Set

$$\psi_{a_2, b_2}(\theta) = \sum_{i,j} [i, j]\theta^{-b_2 i + (a_2 + b_2)j} \tag{6}$$

$$a_2 \not\equiv 0, b_2 \not\equiv 0, a_2 + b_2 \not\equiv 0 \pmod{cl}, i, j = 0, 1, \ldots, cl - 1.$$

We have, as is known (Mitchell[1]),

$$\psi_{a_1, b_2}(\theta)\psi_{a_1, b_2}(\theta^{-1}) = p^t, \tag{7}$$

where $[i, j]$ is defined as before with e replaced by cl, ν by l, and q by p. Set in (6), $b_2 = b_1 l$, $a_2 + b_2 = a_1 c$ with $b_1 \not\equiv 0 \pmod{c}$, $a_1 \not\equiv 0 \pmod{l}$ and this agrees with the restrictions $a_2 \not\equiv 0$, $b_2 \equiv 0$, $a_2 + b_2 \not\equiv 0 \pmod{cl}$. Hence (6) may be written

$$\psi_{a_1, b_2}(\theta) = \sum_{i,j} [i, j]\zeta^{-b_1 l i + a_1 c j}\beta^{-b_1 l i + a_1 c j}$$

or

$$\psi_{a_1, b_2}(\theta) = \sum_{i,j} [i, j]\zeta^{a_1 c j}\beta^{-b_1 l i}.$$

We now write a in lieu of $a_1 c$ and b in lieu of $-b_1 l$ and put

$$\psi_{a_1, b_2}(\theta) = \sum_{i,j} [i, j]\zeta^{aj}\beta^{bi} \tag{8}$$

$$a \not\equiv 0 \pmod{l}, b \not\equiv 0 \pmod{c}.$$

Set $j = r + lh$ with h in the set $0, 1, 2, \ldots, c - 1$, and $0 \leqq r < l$. Then the right-hand member of (8) becomes

$$\sum_r \sum_i \sum_h [i, r]\beta^{bi}\zeta^{ar}.$$

Similarly write $i = v + ck$; $0 \leqq v < c$, $0 \leqq k < l$. Then the above may be written as

$$\sum_r \sum_v \sum_{h=0}^{c-1} \sum_{k=}^{l-1} [v + ck, r + lh]\beta^{bv}\zeta^{ar}.$$

By our[2] theorem on trinomial equations we have

$$\sum_{u=0}^{c-1} \sum_{k=0}^{l-1} [v + ck, r + lh]$$

is the number (v, r) of distinct solutions (g^α, g^γ) of

$$1 + g^{v + c\alpha} = g^{r + l\gamma}$$

α in the set 0, 1, ..., $l - 1$, and γ in the set 0, 1, ..., $c - 1$. Hence (8) gives

$$\psi_{a_1 b_1}(\theta) = \sum_{i,j}(i, j)\beta^{bi}\zeta^{aj} \tag{9}$$

i ranging over the set 0, 1, ..., $c - 1$, and j ranging over the set 0, 1, ..., $l - 1$.

This function may also be written, if i ranges over the values 0, 1, ..., $cl - 1$ excepting $(p^t - 1)/2$,

$$\sum_i \beta^{ib}\zeta^{a \text{ ind } (g^i+1)}.$$

Hence we have the

THEOREM I. *If p is an odd prime belonging to the exponent t modulo l and such that $p^t = 1 + cl$ with l an odd prime and $(c, l) = 1$. Write $\theta = \zeta\beta$, where ζ is a primitive lth root of unity and β is a primitive oth root of unity. Let g be a primitive root of the finite field of order p^t, and denote by (v, r) the number of distinct solutions g^α, g^β of*

$$1 + g^{\beta+c\alpha} = g^{r+l\gamma}$$

α in the set 0, 1, ..., $l - 1$ and γ in the set 0, 1, ... $c - 1$. Associated with this number (v, r) is the cyclotomic function

$$\psi_{a,b}(\theta) = \sum_{i,j}(i, j)\beta^{bi}\zeta^{aj}$$

with $a \not\equiv 0 \pmod{l}$, $b \not\equiv 0 \pmod{c}$, i ranging over the set 0, l, ... $c - 1$ and j ranging over the set 0, 1, ... $l - 1$. Said function has the property

$$\psi_{a,b}(\theta)\psi_{a,b}(\theta^{-1}) = p^t.$$

[1] Mitchell, *Ann. Math.*, **17**, 165–177 (1916).

[2] These PROCEEDINGS, **31**, 170–175; 189–194 (1945).

[3] *Ibid.*, **32**, 51 (1946), Theorem I.

ON SOME SPECIAL TRINOMINAL EQUATIONS IN A FINITE FIELD

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In another paper[1] we obtained results concerning the relations

$$au^e + bv^f + w^g = 0 \tag{1}$$

and

$$au^e + bv^f + 1 = 0, \tag{2}$$

in a finite field, F, of order p^n, designated by $F(p^n)$, where $abuvw \neq 0$, a and b given elements of F. In the case where $n = 1$ explicit expressions were found giving the number of solutions u^e, v^f, w^g in (1) and u^e, v^f in (2). In another paper[2] we considered the equation

$$c_1x_1^{a_1} + c_2x_2^{a_2} \ldots + c_sx_s^{a_s} + c_{s+1} = 0, \tag{3}$$

where the a's are integers such that $0 < a \leqq p^n - 1$; $s \geqq 2$, the c's belong to F; and $c_1, c_2 \ldots c_sx_1x_2 \ldots x_s \neq 0$ in $F(p^n)$. It was there shown (Theorem I) that the number of solutions of (3) may be determined if we know the number of solutions of

$$ky^m + lz^m + 1 = 0, \tag{4}$$

for any given k and l in $F(p^n)$ where $(p^n - 1, a_i) = d_i$; $i = 1, 2, \ldots, s$; and m is the L.C.M. of $d_1, d_2, \ldots, d_s$, provided we have a table of indices for the elements of the field under multiplication. As (1) and (2) are special cases of (3) the method of the second paper applies to each of the former. But the use of the latter method, convenient as it appears, gives us no information in some cases concerning the number of solutions of (1) or (2). This is emphatically brought out by the following investigation. In (1) set $f = g$ and we obtain

$$au^e + bv^f + w^f = 0. \tag{5}$$

further set $ef = p^n - 1$ with $(e, f) = 1$. This is the special case of Theorem I of the first paper herein mentioned, with $f = g$, $ef = p^n - 1$ and $(e, f) = 1$. If we apply the method of the second paper to (5) we note that since the least common multiple of e and f is $p^n - 1$ then the equation (4) reduces to $k + l + 1 = 0$, and this means that we have to know what various equations of this type exist for classes of values k and l, which reduces to the problem we started with. However, the use of another method determines the number exactly as we shall now show.

Consider

$$1 + bx^f \tag{6}$$

where x^f ranges over all e possible values in the field. Each value, unless it is zero in the field, may be expressed in the form:

$$-a\rho^{es}\rho^{ft},$$

since the multiplicative cyclic group of the non-zero elements of the finite field $F(p^n)$, generated by ρ, can be expressed as the product of two cyclic groups of orders e and f, noting that $(e, f) = 1$. If

$$1 + bx^f = -ay^e z^f, \tag{7}$$

say, then it follows that

$$(z^{-1})^f + b(xz^{-1})^f = -ay^e,$$

which has the form (5). Conversely if (5) holds, then

$$1 + b(vw^{-1})^f = -a(w^{-1})^f u^e,$$

which has the form (7). Hence there is a one-to-one correspondence between the possible values u, v and w in (5), and the values x, y and z in (7). Since $ef = p^n - 1$, then there are e solutions of (7), except when there exists an x such that $1 + bx^f = 0$. In the latter case $-bx^f = 1$, and raising each member of this equation to the eth power, we have, since $x^{ef} = x^{p^n-1} = 1$, the condition $(-b)^e = 1$.

Hence we have

THEOREM I. *If $ef = p^n - 1$ $(e, f) = 1$, p is an odd prime then the number of solutions u^e, v^f, w^f, of the equation*

$$au^e + bv^f + w^f = 0; \qquad abuvw \neq 0,$$

with a, b, u, v and w quantities in the finite field of order p^n, a and b given, is e, unless $(-b)^e = 1$ in the field and then the number is $e - 1$.

This is apparently a very special result, but if we apply the methods employed[2] (pp. 48–49) elsewhere we see that the theorem will give the exact number of solutions of the equation given, when e and f are replaced by any of their divisors, so our result has a variety of consequences. In our first paper we proved a Theorem I which gave a direct method for obtaining a residue of the number of solutions of (1) if the present paper modulo p. We were led to Theorem I of the present paper by pursuing this scheme originally. Now we shall write $p^n = 1 + cl$ with l an odd prime and $(c, l) = 1$. To conform to notation used elsewhere, we shall, from now on, use g to denote a multiplicative generator of $F(p^n)$ instead of ρ. If we have an element a of $F(p^n)$ and $g^k = a$ we write $k = \text{ind } a$. Consider the relation

$$g^{i+cs} - 1 = g^{k+ll}, \tag{8}$$

and denote the number of solutions in g^{cs}, g^{ll}, by (i, k). We shall prove that

$$\text{ind } (g^{ll} - 1) \equiv \sum_{h=1}^{l-1} h(i, h), \tag{9}$$

modulo l. From (8) there are $(i, 1)$ solutions g^{cs_1} such that

$$\text{ind } (g^{i+cs_1} - 1) \equiv 1 \pmod{l},$$

and in general there are (i, j) solutions g^{cs_j} such that

$$\text{ind } (g^{i+cs_j} - 1) \equiv j \pmod{l}.$$

Now altogether the values of $s_1, s_2, \ldots, s_{l-1}$ are congruent to the $l - 1$ numbers $1, 2, 3, \ldots, l - 1$, modulo l. Hence

$$\sum_{j=0}^{l-1} \text{ind } (g^{i+cj} - 1) \equiv \sum_{j=0}^{l-1} j(i, j), \tag{10}$$

modulo l. Now we have if A and B are elements, $\neq 0$, of $F(p^n)$

$$\text{ind } A + \text{ind } B \equiv \text{ind } (AB) \pmod{l},$$

and the left-hand member of (10) may then be written

$$\text{ind } (\pi g^{cj} \pi (g^i - g^{-cj})),$$

noting that

$$g^{i+cj} - 1 = g^{-cj}(g^i - g^{-cj}).$$

We have

$$\pi g^{cj} = g^{cln} = 1,$$

where $m = (l - 1)/2$. Also

$$\pi (g^i - g^{cj}) = (g^{ll} - 1),$$

since g^{-cj} satisfies $x^l = 1$ in the field for any j. This gives (9).

In another paper the writer showed that if $(a(a + 1), p) = 1$ then for some k in the set $0, 1, \ldots, l - 1$, we have

$$\sum_{s=0}^{\infty} \binom{cr}{k + sl} a^{k+sl} \equiv 1 \pmod{p},$$

and for any m in the set $0, 1, \ldots, l - 1$ with $m \neq k$ we have

$$\sum_{s=0}^{\infty} \binom{cr}{m + sl} a^{sl} \equiv 0 \pmod{p}.$$

Here

$$\binom{v}{w} = 0,$$

if $w > v$. We shall now extend the ideas in the proof of this by considering the special equation

$$g^{xd} + g^{lc} = g^{rl}, \tag{12}$$

in $F(p^n)$, where $d = cl/(p-1)$, and obtain other congruences involving the number of solutions of (11) and binomial coefficients. We also arrive at a number of congruences involving binomial coefficients alone. Take p to be a primitive root of l so that $p^{l-1} - 1 = lc$. We represent the finite field $F(p^{l-1})$ by means of the set of residue classes with respect to $\mathfrak{p}$, a prime ideal factor of (p) in the algebraic field $k(\zeta)$ where $\zeta = e^{2i\pi/l}$. Since p is a primitive root of l then $\mathfrak{p} = (p)$ and from $(g^{xd})^{p-1} \equiv 1 \pmod{\mathfrak{p}}$ we may replace g^{xd} by a non-zero rational integer, undetermined. Also we may select the generator (primitive root) g to be such that $g^c \equiv \zeta \pmod{\mathfrak{p}}$. Hence, the problem of finding the number of solutions of (12) is equivalent to the problem of finding the rational integers v, t and r such that

$$1 + v\zeta^t \equiv \rho^r \pmod{\mathfrak{p}}, \tag{13}$$

where ρ is a primitive root of the congruence $x^c \equiv 1 \pmod{p}$ in $k(\zeta)$.

We now consider the solutions of (13) with v fixed. There is at least one solution given by $1 + v \equiv \rho_0 \pmod{\mathfrak{p}}$, where $\rho_0^c \equiv 1 \pmod{\mathfrak{p}}$, since $p - 1 \not\equiv 0 \pmod{l}$ and hence any rational integer is congruent to an lth power modulo p. This may be the only solution of (13) for v fixed, but if there is another it must be of the type $1 + v\zeta^a \equiv \rho_a$; $\rho_a^c \equiv 1 \pmod{\mathfrak{p}}$; $a \not\equiv 0 \pmod{l}$. If in the last congruence involving v we use the substitution (ζ^i/ζ), $i = 1, 2, \ldots l-1$ we obtain, since $\mathfrak{p} = (p)$, the relations

$$1 + v\zeta^i \equiv \rho_i, \tag{14}$$

$i = 0, 1, \ldots, l-1$; $\rho_i^c \equiv 1 \pmod{\mathfrak{p}}$. If (14) exists then it is *called an A-set corresponding to* v. Now consider the case where $v = p - 1$ in particular. Here $1 + v \equiv 0$, so that (14) does not exist for $i = 0$, but the other congruences in (14) may hold. If this is the case we say that there is a *B-set corresponding to* $p - 1$. If there is a v, say v_1, which does not satisfy (14) for $i = 1$, then there is just one relation involving v_1, namely

$$1 + v_1 = \rho'; \qquad (\rho')^c \equiv 1 \pmod{\mathfrak{p}}$$

of the type (13). Then there is said to be a *C-set corresponding to* v_1. Now take all the possible values of v in (13), namely $1, 2, \ldots, p-2$, and also $(p-1)$ provided $t \not\equiv 0 \pmod{l}$. Let n be the number of A-sets existing, then there are $p - 2 - n$, C-sets.

We shall derive a criterion for the existence of a B-set. In that case we have

$$1 + (p-1)\zeta^t \equiv \rho; \quad \rho^c \equiv 1$$

$$1 - \zeta^t \equiv \rho (\mathrm{mod}\ \mathfrak{p}),$$

$$\zeta^{1/2}(\zeta^{-1/2} - \zeta^{1/2}) \equiv \rho (\mathrm{mod}\ \mathfrak{p}),$$

and using the substitution (ζ/ζ^{-1}) we have, if $\rho_1^c \equiv 1$,

$$\zeta^{-1/2}(\zeta^{1/2} - \zeta^{-1/2}) \equiv \rho_1 \ (\mathrm{mod}\ \mathfrak{p}),$$

whence

$$-\zeta \equiv \rho/\rho_1 \ (\mathrm{mod}\ \mathfrak{p}),$$

and $\zeta^c \equiv 1 (\mathrm{mod}\ \mathfrak{p})$. And since $\mathfrak{p}$ is prime to l, then $c \equiv 0 (\mathrm{mod}\ l)$. But from the fact that this gives

$$\zeta^{-t/2} - \zeta^{t/2} \equiv \rho' (\mathrm{mod}\ \mathfrak{p})$$

we obtain, if $(\rho')^c \equiv 1$,

$$1 - \zeta^t \equiv \zeta^{t/2}\rho' (\mathrm{mod}\ \mathfrak{p})$$

or

$$1 - (p-1)\zeta^t \equiv \zeta^{t/2}\rho' (\mathrm{mod}\ \mathfrak{p})$$

Hence a necessary and sufficient condition for the existence of a B-set is that $c \equiv 0 (\mathrm{mod}\ l)$.

We now assume that $(c, l) = 1$ so that no B-set exists. In view of this there exist n, A-sets and $p - 2 - n$, C-sets so that the number of solutions N_0 in v, ζ, ρ, of (13) is

$$N_0 = p - 2 - n + nl. \tag{15}$$

We now proceed to find another expression for N_0. Using the method in another paper[1] we have where now i ranges over 0, 1, . . ., $p - 2$,

$$\sum_{i,\,t,\,s} (1 - (1 + h^i\zeta^t - \rho^s)^{p^l-1-1}) \equiv N, \tag{16}$$

modulo $\mathfrak{p}$, where h is a primitive root of p in the rational field, t ranges over the set 0, 1, . . ., $l - 1$, and s over the set 0, 1, . . ., $c - 1$. This reduces, using the methods of the same paper, modulo p, to

$$N_0 \equiv -\sum_{w=0}^{\infty} \sum_{k=1}^{l-1} \binom{kc}{ul(p-1)}. \tag{17}$$

In the same way we reduce the expression

$$N_e = \sum_{i\,t\,s} (\zeta^e - \zeta^e(1 + h^i\zeta^t - \rho^s)^{p^l-1-1}),$$

where i, t and s range as in (16), which is congruent modulo $\mathfrak{p}$ to the sum of the eth powers of all the possible values of ζ in v, ζ, ρ, of (13), modulo p to

$$N_e \equiv -\sum_{k=1}^{l-1}\sum_{u=0}^{\infty}\binom{kc}{s_e + ul(p-1)}, \tag{17a}$$

where s_e is selected so that $s_e \equiv 0 \pmod{p-1}$ and also $s_e \equiv -e \pmod{l}$ for any e such that $0 \leqq e < l$. But if $e \not\equiv 0 \pmod{l}$ the sum of the e-th powers of the solutions ζ^t in (13) is easily seen to be $p - 2 - n$, so that

$$N_e \equiv -2 - n \pmod{p}, \tag{18}$$

$e = 1, 2, \ldots, l-1$. Eliminating n from the congruences (15) and (18) gives

$$N_0 + (l-1)N_e \equiv -2l \pmod{p} \tag{19}$$

$e \not\equiv 0 \pmod{l}$.

We shall now find some other relations involving N_0. The expression $1 + v\zeta^t$, as v ranges over $1, 2, \ldots, p-1$ and t over $0, 1, \ldots, l-1$ independently can in general be expressed in the form $\zeta^\tau\rho$, where $\rho^c \equiv 1$, provided $(c, l) = 1$. Hence we have

$$(1 + v\zeta^t)^{fc} \equiv \zeta^d \pmod{\mathfrak{p}}. \tag{20}$$

Assume $f \not\equiv 0 \pmod{l}$. Then let v and t range as above indicated and add together the resulting congruences. Corresponding to each v for which an A-set exists the sum for these congruences on the right is l, so for all such v's the sum on the right is nl. For a v such that we have a C-set the quantities on the right are $1, \zeta, \zeta^2, \ldots, \zeta^{l-1}$ in some order and the sum is zero. The other possibility is when $v = p - 1$ and then the sum on the right for this value of v is (-1). When we add the congruences in (20) on the left for the various values of ζ and t each term involving these is zero except those whose binomial coefficients have the form

$$\binom{fc}{ul(p-1)},$$

and we then find

$$(p-1)l\sum_{u=0}^{\infty}\binom{fc}{ul(p-1)} \equiv nl - 1,$$

modulo $\mathfrak{p}$. Since we have noted that (12) and (13) are equivalent, and using (15) we then obtain

Theorem II. *If p is a prime and a primitive root of the prime l, $p^{l-1} - 1 = lc$, $(l, c) = 1$, g a primitive root of the finite field $F(p^{l-1})$, then the number of solutions in g^s, g^t, g^r of the equation*

$$g^{ns} + g^{tc} = g^{rl}$$

in $F(p^{l-1})$, where $d = cl/(p-1)$, is $p - 2 + (l-1)n$, where n is the least residue $\geqq 0$ of

$$-c - B_f,$$

modulo p, and where

$$B_f = \sum_{u=0}^{\infty} \left(\frac{fc}{ul(p-1)} \right).$$

Also, modulo p,

$$B_1 \equiv B_a; \qquad a = 1, 2, \ldots, l-1.$$

[1] These PROCEEDINGS, **31**, 170–172 (1945).
[2] *Ibid.*, **32**, 47–52 (1946).

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*A PARTIAL DIFFERENTIAL EQUATION OF FOURTH ORDER CONNECTED WITH RATIONAL FUNCTIONS OF A COMPLEX VARIABLE**

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1. We shall consider the class of polynomials $\phi(x, y)$ which are obtained as the numerator of the real part of a rational function of a complex variable $z = x + iy$. This new class contains the harmonic polynomials as a proper subset. In general, a polynomial $\phi(x, y)$ of this new class does not satisfy the Laplace equation. We find that it does obey a certain partial differential equation of fourth order. Essentially we are dealing with the numerators of all rational solutions of the Laplace equation.

2. Harmonic polynomials are well known.[1] These are obtained as the real part of a polynomial or rational integral function in z. They constitute the class of polynomial solutions $\phi(x, y)$ of the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0. \tag{1}$$

3. Consider a rational function $w = f(z)$ of the complex variable z. This is expressed as the quotient of two relatively prime polynomials in z. The degree r of $f(z)$ is the maximum of the two degrees of the numerator and denominator.

Upon decomposing this rational function $f(z)$ into its real and imaginary parts, we see that it can be written in the form

$$f(z) = \frac{\phi(x, y)}{D(x, y)} + i\frac{\psi(x, y)}{D(x, y)}, \tag{2}$$

where $\phi(x, y)$, $\psi(x, y)$, $D(x, y)$ are real polynomials in the real variables (x, y) such that the numerator and denominator of each of the fractions appearing in the above equation have no common factors. Of course, $D(x, y)$ can vanish only at the poles of $f(z)$.

The degree of $\phi(x, y)$ (or $\psi(x, y)$ or $D(x, y)$) is $2r - k$ where $0 \leqq k \leqq r$.

The new class of polynomials that we wish to study are all the polynomials $\phi(x, y)$(or $\psi(x, y)$) which are obtained in this way.[2] Of course each of the component fractions in equation (2) satisfies the Laplace equation, but *the polynomials* $\phi(x, y)$, $\psi(x, y)$, $D(x, y)$, *do not obey the Laplace equation in general.* Obviously the harmonic polynomials are a proper subset of our new class.

4. FUNDAMENTAL THEOREM. *A polynomial $\phi(x, y)$ is the numerator of the real (or imaginary) part of a rational function of a complex variable if and only if it obeys the partial differential equation of fourth order*

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \log\left[\phi(\phi_{xx} + \phi_{yy}) - (\phi_x^2 + \phi_y^2)\right] = 0. \tag{3}$$

In expanded form, this is

$$\begin{aligned}[\phi(\phi_{xx} + \phi_{yy}) - (\phi_x^2 + \phi_y^2)][\phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy}] \\ - \phi[(\phi_{xxx} + \phi_{xyy})^2 + (\phi_{xxy} + \phi_{yyy})^2] + 2\phi_x[(\phi_{xx} - \phi_{yy})(\phi_{xxx} + \phi_{xyy}) + \\ 2\phi_{xy}(\phi_{xxy} + \phi_{yyy})] + 2\phi_y[2\phi_{xy}(\phi_{xxx} + \phi_{xyy}) \\ - (\phi_{xx} - \phi_{yy})(\phi_{xxy} + \phi_{yyy})] - (\phi_{xx} + \phi_{yy})[(\phi_{xx} - \phi_{yy})^2 + 4\phi_{xy}^2] = 0. \end{aligned} \tag{4}$$

Moreover, the bracketed expression of second order in (3) is not zero, and is always negative for real polynomials $\phi(x,y)$.

The denominator $D(x, y)$ satisfies this partial differential equation. As a matter of fact, it annuls the expression of second order appearing in (3). That is, $D(x, y)$ is any polynomial solution of the partial differential equation of second order

$$\phi(\phi_{xx} + \phi_{yy}) - (\phi_x^2 + \phi_y^2) = 0. \tag{5}$$

5. *The polynomials $\phi(x, y)$ and $\psi(x, y)$ are the numerators of the real and imaginary parts of a rational function of a complex variable if and only if they obey the system of partial differential equations of second order*

$$\left.\begin{aligned}\phi(\phi_{xx} + \phi_{yy}) - (\phi_x^2 + \phi_y^2) &= \psi(\psi_{xx} + \psi_{yy}) - (\psi_x^2 + \psi_y^2), \\ \psi(\phi_{xx} + \phi_{yy}) + \phi(\psi_{xx} + \psi_{yy}) &= 2(\phi_x\psi_x + \phi_y\psi_y).\end{aligned}\right\} \tag{6}$$

In general, the two families of curves $\phi(x, y) = \text{const.}$ and $\psi(x, y) = \text{const.}$ are not orthogonal. However, the two algebraic curves $\phi(x, y) = 0$ and $\psi(x, y) = 0$ intersect orthogonally in $[r^2 + (r - k)^2]$ points.

6. *The real and imaginary parts of the special class (M) of polygenic functions defined as the product of an analytic function of $z = x + iy$ by*

another independent analytic function of $\bar{z} = x - iy$, *constitute the totality of analytic solutions of our partial differential equation* (3) *of fourth order.*

We observe that for this class (M) of polygenic functions, the equations (6) play the rôle of the Cauchy-Riemann equations and the partial differential equation (3) or (4) of fourth order is the analogue of the Laplace equation.

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[1] The curves defined by setting a harmonic polynomial equal to zero were studied by Briot and Bouquet, Bôcher, and Kasner. See Briot and Bouquet, *Theorie des fonctions elliptiques*, Book IV, Chapter II, p. 226 (1875). Kasner, "On the Algebraic Potential Curves," *Bull. Amer. Math. Soc.*, **7**, 392–399 (1901). Also "Some Properties of Potential Surfaces," *Ibid.*, **8**, 243–248 (1902).

[2] The curves obtained by setting any polynomial $\phi(x, y)$ of this class equal to zero, are being studied by the authors. See Kasner and De Cicco, "Rational Functions of a Complex Variable and Related Algebraic Potential Curves," these PROCEEDINGS, **32**, 280–282 (1946). See abstracts in *Bull. Amer. Math. Soc.*, 1946–1947.

A REDUCTION THEOREM CONCERNING THE REPRESENTATION PROBLEM FOR FRÉCHET VARIETIES

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The object of this note is to prove a reduction theorem which will be employed in a forthcoming paper to solve the representation problem for a wide class of Fréchet surfaces. The reduction theorem itself is true in ultimate generality within the class of Fréchet varieties; consequently the discussion here is not restricted to Fréchet surfaces.

A first obligation is to define the terminology employed. If $f_1(X^1) \subset Y \supset f_2(X^2)$ are mappings (= continuous transformations) from Peano spaces X^1 and X^2, then $f_1(X^1)$ is said to be *Fréchet equivalent* to $f_2(X^2)$ if and only if, for every $\epsilon > 0$ there is a homeomorphism $h_\epsilon(X^1) = X^2$ such that $\rho\{f_1(x^1), f_2h_\epsilon(x^1)\} < \epsilon$, for $x^1 \in X^1$. (Notation: $f_1 \sim f_2$.) This equivalence relation partitions the totality of mappings from Peano spaces into equivalence classes $[f]$. Each equivalence class $[f]$ is known as a *Fréchet variety* V, while any mapping in $[f]$ is said to be a *representation* of the Fréchet variety V. It is clear that if $f_1(X^1)$ and $f_2(X^2)$ are representations of the same Fréchet variety V, then: (i) the range spaces X^1 and X^2 are topologically equivalent, (ii) the image spaces $f_1(X^1)$ and $f_2(X^2)$ are identical. Hence with each Fréchet variety V there is associated a pair of Peano spaces: the first, $\tilde{V}$, is topologically equivalent to the range space of any representation of V; the second, $|V|$, is the common image

space of any representation of V. (The topological properties of $\tilde{V}$ serve to catalogue Fréchet varieties; for example, if $\tilde{V}$ is a 2-manifold, M, then V is said to be a Fréchet *surface* of the topological type of M.)

The *representation problem* for Fréchet varieties is the problem of determining the totality of representations of a Fréchet variety V upon being given any one representation, f, of V. In other words, the problem is to determine a *criterion* which will serve to obtain the totality of solutions, g, of the relation $g \sim f$. Such a criterion will be called an F-criterion. The importance of the problem is due to the fact that given one representation of a Fréchet variety there may be more favorable representations.

The definition of Fréchet equivalence itself, of course, serves as an F-criterion, but in applications it is often impossible to decide on the basis of the definition alone whether two given mappings are equivalent or not. Hence, the problem really asks for an effective solution, effective in the sense of its applicability.

For a short history of the problem one may consult a paper by Youngs.[1] Suffice it to say that the only solution to date is for the case in which V is of the topological type of a 2-sphere; that is, $\tilde{V}$ is a 2-sphere. This note serves the purpose of recording a *reduction theorem* in the sense that the problem of finding an F-criterion for general mappings is reduced to the problem of finding an F-criterion for monotone mappings.

The statement of the reduction theorem requires a factor theorem due to Eilenberg and Whyburn. For a proof of the theorem, and the definitions of the terms involved, one may consult Whyburn,[2] pages 141–143.

FACTOR THEOREM. *If $f(X) = Y$ is a mapping from a Peano space, then there is a monotone mapping $m(X) = \mathfrak{X}$ and a light mapping $l(\mathfrak{X}) = Y$ such that $f(x) = lm(x)$, $x \in X$.*

A *monotone-light factorization* of $f(X)$ is simply a factorization of the above type. (If $f(X) = l_1m_1(X)$, $m_1(X) = \mathfrak{X}^1$ and $f(X) = l_2m_2(X)$, $m_2(X^2) = \mathfrak{X}^2$ are two monotone-light factorizations, then there is a unique homeomorphism $h(\mathfrak{X}^1) = \mathfrak{X}^2$ such that $hm_1(x) = m_2(x)$, $x \in X$, and $l_2h(\mathfrak{x}^1) = l_1(\mathfrak{x}^1)$, $\mathfrak{x}^1 \in \mathfrak{X}^1$.)

REDUCTION THEOREM. *Two mappings $f_1(X^1)$ and $f_2(X^2)$ from Peano spaces X^1 and X^2 are Fréchet equivalent if and only if there are monotone-light factorizations $f_1(X^1) = lm_1(X^1)$ and $f_2(X^2) = lm_2(X^2)$ such that $m_1(X^1)$ and $m_2(X^2)$ are Fréchet equivalent.*

Proof. Suppose first that there are monotone-light factorizations of the above type. Then $m_1(X^1) = \mathfrak{X} = m_2(X^2)$, and as l is continuous on the Peano space $\mathfrak{X}$ it is uniformly continuous. Hence, for any $\epsilon > 0$ there is a δ such that $\rho\{\mathfrak{x}_1, \mathfrak{x}_2\} < \delta$ implies $\rho\{l(\mathfrak{x}_1), l(\mathfrak{x}_2)\} < \epsilon$. Since $m_1 \sim m_2$ there is a homeomorphism $h_\delta(X^1) = X^2$ with the property that $\rho\{m_1(x^1), m_2h_\delta(x^1)\} < \delta$, $x^1 \in X^1$. Therefore, $\rho\{f_1(x^1), f_2h_\delta(x^1)\} = \rho\{lm_1(x^1), lm_2h_\delta(x^1)\} < \epsilon$, $x^1 \in X^1$, and so $f_1 \sim f_2$.

The converse follows as a direct consequence of a theorem of Radó[3] (page 425) or Youngs[1] (page 714). The development of Radó is to be preferred and will be used here. The basic statement is the following:

LEMMA OF RADÓ. *If* $lg_n(X)$, $g_n(X) = \mathfrak{X}$, $n = 1, 2, 3, \ldots$ *is a sequence of equicontinuous mappings from a Peano space* X *and the mapping* l *is light, then* $\{g_n\}$ *is also a sequence of equicontinuous mappings.*

Since $f_1 \sim f_2$, for each n there is a homeomorphism $h_n(X^1) = X^2$ such that $\rho\{f_1(x^1), f_2h_n(x^1)\} < 1/n$, $x^1 \in X^1$. In other words, $f_2h_n(x^1) \rightrightarrows f_1(x^1)$, $x^1 \in X^1$ ($= f_2h_n(x^1)$ converges uniformly to $f_1(x^1)$ on X^1). Let $f_2(X^2) = lm_2(X^2)$, $m_2(X^2) = \mathfrak{X}$ be a monotone-light factorization. Now $lm_2h_n(x^1) \rightrightarrows f_1(x^1)$, $x^1 \in X^1$, and so $\{lm_2h_n\}$ is a sequence of equicontinuous mappings. Consequently, by the lemma of Radó quoted above, $\{m_2h_n\}$ is also a sequence of equicontinuous mappings. Since X is Peanian, the sequence $\{m_2h_n\}$ contains a uniformly convergent subsequence, and it may be assumed that the notation $\{m_2h_n\}$ refers to this subsequence, while m_1 denotes its limit. It is important to notice that each mapping m_2h_n is monotone, consequently the statement $m_2h_n(x^1) \rightrightarrows m_1(x^1)$, $x^1 \in X^1$ has several immediate implications.

(i) $m_1 \sim m_2$.

(ii) $m_2h_n(X^1) = \mathfrak{X}$, hence $m_1(X^1) = \mathfrak{X}$ is monotone by a theorem of Whyburn[2] (page 174).

(iii) $lm_2h_n(x^1) \rightrightarrows lm_1(x^1)$, $x^1 \in X^1$. On the other hand, $lm_2h_n(x^1) = f_2h_n(x^1) \rightrightarrows f_1(x^1)$, $x^1 \in X^1$, and so $f_1(x^1) = lm_1(x^1)$, $x^1 \in X^1$.

This shows that there are monotone-light factorizations $f_1(X^1) = lm_1(X^1)$ and $f_2(X^2) = lm_2(X^2)$ such that $m_1 \sim m_2$.

In conclusion it should be explicitly stated that the argument used in the second half of this theorem is the argument employed by Radó[3] (page 426). The only conclusion drawn here which is not mentioned by Radó is that $m_1 \sim m_2$. On the other hand, Radó proves a simple but fundamental statement (not recorded here) on the basis of which he is able to prove the most general theorems known concerning the cyclic additivity of the Lebesgue and de Geöcze areas of a Fréchet surface—a beautiful result. In view of these comments it is not unlikely that this reduction theorem, which will be of fundamental importance in the proposed solution of the representation problem, has apparently escaped mention because there was no intention to apply it. It is exhibited here as the key to a proposed body of research.

* Fellow of the John Simon Guggenheim Memorial Foundation.

[1] Youngs, J. W. T., "The Topological Theory of Fréchet Surfaces," *Annals Math.*, **45**, 753–785 (1944).

[2] Whyburn, G. T., "Analytic Topology," American Mathematical Society Colloquium Publications, Vol. 28 (1942).

[3] Radó, Tibor, "On Continuous Mappings of Peano Spaces," *Trans. Amer. Math. Soc.*. **58**. 420–454 (1945).

ON SETS OF INTEGERS WHICH CONTAIN NO THREE TERMS IN ARITHMETICAL PROGRESSION

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Let S be a set of non-negative integers $\leq N$ no three of which form an arithmetical progression (i.e., $A + A' \neq 2A''$ for any three distinct terms of S). Let $r(N)$ denote the maximum number of terms of such a "progression-free" set. Salem and Spencer[1] proved that for $\epsilon > 0$ and sufficiently large N

$$r(N) > N^{1 - \frac{\log 2 + \epsilon}{\log \log N}}.$$

I will show in this note that, by a modification of their method, the better estimate

$$r(N) > N^{1 - \frac{2\sqrt{2 \log 2} + \epsilon}{\sqrt{\log N}}}$$

can be obtained.

For any integers $d \geqslant 2$, $n \geqslant 2$, $k \leq n(d-1)^2$ consider the set $S_k(n, d)$ of all numbers of the form

$$A = a_1 + a_2(2d-1) + \ldots + a_n(2d-1)^{n-1}$$

where the "digits" a_i are integers subject to the conditions

$$0 \leq a_i < d \tag{i}$$

$$(\text{norm } A)^2 = k \tag{ii}$$

where

$$\text{norm } A = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}.$$

This set is progression-free; for suppose $A + A' = 2A''$ for A, A', A'' in $S_k(n, d)$ then

$$\text{norm } (A + A') = \text{norm } (2A'') = 2\sqrt{k}$$

and

$$\text{norm } A + \text{norm } A' = 2\sqrt{k}.$$

Thus, in the triangular inequality

$$\text{norm } (A + A') \leq \text{norm } A + \text{norm } A'$$

equality holds which is only possible if $(a_1, a_2, \ldots, a_n)$ and $(a_1', a_2', \ldots, a_n')$ are proportional and, as their norms are equal, identical, i.e., if $A = A' = A''$.

There are d^n different systems $(a_1, a_2, \ldots, a_n)$ satisfying (i) and $n(d-1)^2+1$ possible values of k; hence for some $k = K$, $S_k(n, d)$ must contain at least

$$\frac{d^n}{n(d-1)^2+1} > \frac{d^{n-2}}{n}$$

terms; as all these terms are $< (2d-1)^n$ we have

$$r((2d-1)^n) > \frac{d^{n-2}}{n}.$$

Let N be given; choose $n = \left[\sqrt{\frac{2\log N}{\log 2}}\right]$, and d such that

$$(2d-1)^n \leq N < (2d+1)^n.$$

Then,

$$r(N) \geqslant r((2d-1)^n) > \frac{d^{n-2}}{n} > \frac{(N^{1/n}-1)^{n-2}}{n2^{n-2}} = \frac{N^{1-(2/n)}}{n2^{n-2}}(1-N^{-1/n})^{n-2},$$

and, for sufficiently large N,

$$r(N) > \frac{N^{1-(2/n)}}{n2^{n-1}} = N^{1-\frac{2}{n}-\frac{\log n}{\log N}-\frac{(n-1)\log 2}{\log N}} > N^{1-\frac{2\sqrt{2\log 2}+\epsilon}{\sqrt{\log N}}}$$

for any $\epsilon > 0$.

[1] Salem, R., and Spencer, D. C., "On Sets of Integers Which Contain No Three Terms in Arithmetical Progression," these PROCEEDINGS, **28**, 561–563 (1942).

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